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ABSTRACT

Numerical approximations and a Fortran IV program are given for the calculation by an IBM 7090 computer of the complete and incomplete elliptic integrals of the third kind. In its present form results are limited to six decimal places, but the method is valid for all values of amplitude φ, modulus k and real values of the parameter α^2. For the purpose of completeness, adaptations of other programs for complete and incomplete elliptic integrals of the first and second kind are also presented.

TABLE OF CONTENTS

LIST OF SYMBOLS AND DEFINITIONS

α^2 — parameter in the elliptic integral of the third kind.

k, k' — modulus and complementary modulus, $0 \leqslant k^2 \leqslant 1$ and $k' = \sqrt{1 - k^2}$.

y or φ — argument of the elliptic integral $0 < y \leqslant 1$, $0 < \varphi \leqslant \frac{\pi}{2}$.

$F(\varphi, k)$, $E(\varphi, k)$, $\Pi(\varphi, \alpha^2, k)$ — elliptic integrals of the first, second and third kind respectively, where

$$F(\varphi, k) = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}}$$

$$E(\varphi, k) = \int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\, d\theta$$

and

$$\Pi(\varphi, \alpha^2, k) = \int_0^y \frac{1}{1-\alpha^2t^2} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$= \int_0^\varphi \frac{1}{1-\alpha^2\sin^2\theta} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} .$$

K, E and $\Pi(\alpha^2, k)$ — complete elliptic integrals of the first, second and third kind respectively, where

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$E = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta$$

$$\Pi(\alpha^2, k) = \int_0^1 \frac{1}{1-\alpha^2t^2} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$= \int_0^{\pi/2} \frac{1}{1-\alpha^2\sin^2\theta} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} .$$

K^*, E^*	approximation functions of complete elliptic integrals of the first and second kind.
$Z(\varphi, k)$	Jacobian Zeta Function. $$Z(\varphi, k) = E(\varphi, k) - \frac{E}{K} F(\varphi, k)$$
$\Lambda_o(\varphi, k)$	Heuman's Lambda Function. $$\Lambda_o(\varphi, k) = \frac{2}{\pi} \{E F(\varphi, k') + K[E(\varphi, k') - F(\varphi, k')]\}$$

1. Introduction

There are tables of the elliptic integral of the third kind, [1], [2] *; these tables are somewhat tedious to use because of the limited number of entries and range necessary in a three parameter tabulation.** Either cross plotting or non-linear interpolation are necessary to obtain values of Π within the range of the tables and furthermore, values of the integral outside the range of the tables must be calculated from cumbersome addition formulae as well as interpolation. These methods are undesirable when very many values of these integrals are needed and completely impractical for adaptation to computer use.

Our interest and need of evaluating these elliptic integrals arise in a study of free-streamline potential flows. This problem requires the numerical integration of expressions involving Π as well as evaluating all of the elliptic integrals many times. Since these calculations are performed on an IBM 7090 digital computer, it is necessary to have computer methods of evaluating these elliptic integrals as needed.

Three subroutines have been written in Fortran IV language to accomplish this result: the complete elliptic integrals of the first and second kind K and E; the incomplete elliptic integrals of the first and second kind, $F(\varphi, k)$ and $E(\varphi, k)$; and most especially the complete and incomplete elliptic integrals of the third kind, $\Pi(\alpha^2, k)$ and $\Pi(\varphi, \alpha^2, k)$

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* The numbers in brackets refer to references at the end of the text.

** In Ref. [1] for example, $-1 < \alpha^2 < 1$ and in Ref. [2] $-1 < \alpha^2 < 0$.

respectively. The first two of these programs closely follow methods outlined by others, Refs. [3] and [4] and are detailed in Appendices A and B respectively.

The present programs are written in single precision arithmetic since that accuracy is sufficient for our calculations. The accuracy of the program for K and E, which uses the approximate formulae given in Ref. [3], would not be improved by converting to double precision arithmetic. The remaining subroutines use series representations and consequently their accuracy can be improved by carrying more significant figures. All programs are, in their present form, accurate to six figures with the following exceptions: when $k \sin\varphi > 0.9$, $E(\varphi, k)$ and $F(\varphi, k)$ lose accuracy; when $k^2 > 0.9$ or $|\alpha^2 - 1| < 0.1$, $\Pi(\varphi, \alpha^2, k)$ and $\Pi(\alpha^2, k)$ lose accuracy.* Four figure accuracy is guaranteed except $k \sin\varphi > 0.9999$ or $|1 - \alpha \sin\varphi| < 0.0001$. Whenever the accuracy is less than six figures, comments are printed and indicators are set to inform the user of this loss in accuracy.

The flow chart and the Fortran IV program listings are included in Appendix C.

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* This result is because the accuracy in computing $(1 - k \sin\varphi)$, $(1 - \alpha^2)$, and $(1 - \alpha \sin\varphi)$ decreases as α and k approach unity as φ tends to $\pi/2$.

2. Complete Elliptic Integrals of the Third Kind

The complete elliptic integral of the third kind, i. e.,

$$\Pi(\alpha^2, k) = \int_0^1 \frac{1}{1-\alpha^2 t^2} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \int_0^{\pi/2} \frac{1}{1-\alpha^2 \sin^2\theta} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} \tag{1}$$

depends on the two parameters α^2 and k. Considering only real values of α^2, the ranges of these parameters are

$$0 \leqslant k^2 \leqslant 1 \quad \text{and} \quad -\infty < \alpha^2 < \infty .$$

Some special cases of α^2 and k are as follows:

a) k = 0; the integral can be integrated in closed form and is

$$\Pi(\alpha^2, 0) = \begin{cases} \dfrac{\pi}{2} \dfrac{1}{\sqrt{1-\alpha^2}} , & \alpha^2 < 1 \\ \infty , & \alpha^2 = 1 \\ 0^* , & \alpha^2 > 1 \end{cases} \tag{2}$$

b) $k = 1$; $\qquad \Pi(\alpha^2, 1) = \infty \qquad (3)$

c) $\alpha^2 = 0$; $\qquad \Pi(0, k) = K \qquad (4)$

d) $\alpha^2 = \pm k$; $\qquad \Pi(\pm k, k) = \dfrac{1}{4(1 \mp k)} [\pi + 2(1 \mp k) K] \qquad (5)$

e) $\alpha^2 = k^2$; $\qquad \Pi(k^2, k) = \dfrac{E}{1-k^2} \qquad (6)$

f) $\alpha^2 = 1$; $\qquad \Pi(1, k) = \infty \qquad (7)$

The remainder of the values of α^2 and $0 < k^2 < 1$ can be classified into four cases as shown in Fig. 1.

---

* For $\alpha^2 > 1$, the integral is interpretated by its Cauchy's principal value.

$$\left.\begin{array}{ll}\text{Case I,} & 0 < -\alpha^2 < \infty \\ \text{Case II,} & k^2 < \alpha^2 < 1\end{array}\right\} \text{Circular cases;}$$

$$\left.\begin{array}{ll}\text{Case III,} & 0 < \alpha^2 < k^2 \\ \text{Case IV,} & 1 < \alpha^2 < \infty\end{array}\right\} \text{Hyperbolic cases;}$$

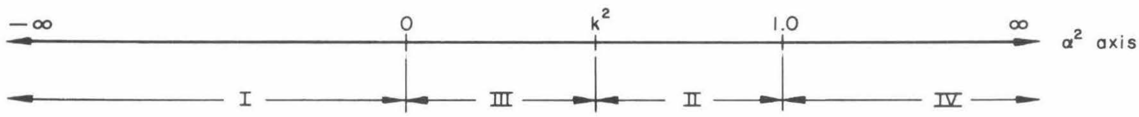

Figure 1

The integral, $\Pi(\alpha^2, k)$, can be evaluated in terms of elementary functions and Heuman's Lambda function*

$$\Lambda_0(\varphi, k) = \frac{2}{\pi}\{E\,F(\varphi, k') + K[E(\varphi, k') - F(\varphi, k')]\} \tag{8}$$

for the circular cases (I and II) or of the Jacobian Zeta function

$$Z(\varphi, k) = E(\varphi, k) - \frac{E}{K}\,F(\varphi, k) \tag{9}$$

or

$$K\,Z(\varphi, k) = K\,E(\varphi, k) - E\,F(\varphi, k) \tag{9a}$$

for the Hyperbolic cases (III and IV).

* These definitions and formulae can be found in Ref. [5].

The Lambda function and Zeta function as given in Eqs. (8) and (9a) are tabulated [5] or can readily be evaluated by using the methods described in Appendices A and B to calculate K, E, $F(\varphi, k)$ and $E(\varphi, k)$.

Case I: $\underline{0 < -\alpha^2 < \infty}$

$$\Pi(\alpha^2, k) = \frac{k^2}{k^2 - \alpha^2} K - \frac{\pi}{2} \frac{\alpha^2 \Lambda_0(\psi, k)}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}} \tag{10}$$

where

$$\psi = \sin^{-1}\sqrt{\frac{\alpha^2}{\alpha^2 - k^2}} \tag{11}$$

or

$$\Pi(\alpha^2, k) = \frac{K}{1 - \alpha^2} + \frac{\pi}{2} \frac{\alpha^2[\Lambda_0(\beta, k) - 1]}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}} \tag{12}$$

where

$$\beta = \sin^{-1}\sqrt{\frac{1}{1 - \alpha^2}} \quad . \tag{13}$$

It is obvious that less time is required to evaluate $F(\varphi, k)$ and $E(\varphi, k)$ when φ is small. Since

$$\psi < \beta \quad \text{when} \quad -k < \alpha^2 < 0$$

and

$$\psi > \beta \quad \text{when} \quad -\infty < \alpha^2 < -k$$

$\Pi(\alpha^2, k)$ is programmed by Eq. (10) in the first interval and by Eq. (12) in the second interval.

Case II: $\underline{k^2 < \alpha^2 < 1}$

$$\Pi(\alpha^2, k) = K + \frac{\pi}{2} \frac{\alpha[1 - \Lambda_0(\theta, k)]}{\sqrt{(\alpha^2 - k^2)(1 - \alpha^2)}} \tag{14}$$

where

$$\theta = \sin^{-1}\sqrt{\frac{1 - \alpha^2}{1 - k^2}} \tag{15}$$

or

$$\Pi(\alpha^2, k) = \frac{\pi}{2} \frac{\alpha \Lambda_0(\xi, k)}{\sqrt{(\alpha^2 - k^2)(1 - k^2)}} \tag{16}$$

where

$$\xi = \sin^{-1} \sqrt{\frac{\alpha^2 - k^2}{\alpha^2 (1 - k^2)}} \quad . \tag{17}$$

In the ranges

$$k < \alpha^2 < 1 \quad , \quad \theta < \xi$$

and when

$$k^2 < \alpha^2 < k \quad , \quad \theta > \xi \quad .$$

In the program, Eq. (14) is used for $\theta < \xi$ and Eq. (16) is used for $\theta > \xi$.

Case III: $\underline{0 < \alpha^2 < k^2}$

$$\Pi(\alpha^2, k) = K + \frac{\alpha K Z(\beta, k)}{\sqrt{(1 - \alpha^2)(k^2 - \alpha^2)}} \tag{18}$$

where

$$\beta = \sin^{-1} \frac{\alpha}{k} \quad . \tag{19}$$

Case IV: $\underline{1 < \alpha^2 < \infty}$

$$\Pi(\alpha^2, k) = - \frac{\alpha K Z(A, k)}{\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}} \tag{20}$$

where

$$A = \sin^{-1} \frac{1}{\alpha} \quad . \tag{21}$$

The flow chart and the subroutine programmed in Fortran IV language* are presented in Appendix C.

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* All programs in this report were written for the IBM 7090 Computer at Booth Computing Center, California Institute of Technology.

3. Incomplete Elliptic Integrals of the Third Kind

The incomplete elliptic integral of the third kind

$$\Pi(\varphi, a^2, k) = \int_0^{\varphi} \frac{1}{1 - a^2 \sin^2\theta} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

$$= \int_0^{y} \frac{1}{1 - a^2 t^2} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{22}$$

has one more parameter than the complete one and consequently is more complicated. In general, $\Pi(\varphi, a^2, k)$ can be classified as circular or hyperbolic, in the same way as the complete integral, however, in our subroutine, a different and more straightforward approach is taken.

First, we shall consider the special cases which can be integrated in terms of elementary functions and $F(\varphi, k)$ and $E(\varphi, k)$.

a) $\varphi = \frac{\pi}{2}$; $\Pi(\frac{\pi}{2}, a^2, k) = \Pi(a^2, k)$ (23)

b) $a^2 = 0$; $\Pi(\varphi, 0, k) = F(\varphi, k)$ (24)

c) $k^2 = 0$;

$$\Pi(\varphi, a^2, 0) = \begin{cases} \varphi & a^2 = 0 \quad (25a) \\[2ex] \dfrac{1}{\sqrt{1-a^2}} \tan^{-1}\left[\sqrt{1-a^2}\tan\varphi\right] & a^2 < 1 \quad (25b) \\[2ex] \dfrac{1}{2\sqrt{a^2-1}} \log\left|\dfrac{\sqrt{a^2-1}\tan\varphi + 1}{\sqrt{a^2-1}\tan\varphi - 1}\right| & a^2 > 1 \quad (25c) \\[2ex] \tan\varphi & a^2 = 1 \quad (25d) \end{cases}$$

d) $k = 1$;

$$\Pi(\varphi, a^2, 1) = \begin{cases} \dfrac{\sin\varphi}{2\cos\varphi} + \dfrac{1}{4}\log\left[\dfrac{1+\sin\varphi}{1-\sin\varphi}\right] & a^2 = 1 \quad (26a) \\[2ex] \dfrac{1}{1-a^2}\left[\log(\tan\varphi + \sec\varphi) - \dfrac{a}{2}\log\left|\dfrac{1+a\sin\varphi}{1-a\sin\varphi}\right|\right] & a^2 > 0,\ a^2 \neq 1 \quad (26b) \\[2ex] \dfrac{1}{1-a^2}\left[\log(\tan\varphi + \sec\varphi) + \sqrt{|a^2|}\tan^{-1}\left(\sqrt{|a^2|}\sin\varphi\right)\right] & a^2 < 0 \quad (26c) \end{cases}$$

e) $a^2 = 1$;

$$\Pi(\varphi, 1, k) = \left[k'^2 F(\varphi, k) - E(\varphi, k) + \tan\varphi\sqrt{1 - k^2\sin^2\varphi}\right]k'^{-2} \tag{27}$$

f) $a^2 = k^2$;

$$\Pi(\varphi, k^2, k) = \left[E(\varphi, k) - \frac{k^2 \sin\varphi\cos\varphi}{\sqrt{1 - k^2\sin^2\varphi}}\right]k'^{-2} \tag{28}$$

The remaining values of a^2, for $0 < k^2 < 1$, and $0 < \varphi < \pi/2$ can be divided into three regions, with two series expansions [5]. In the regions

$$|a^2| > 1,\ \ k^2 < 1 \quad \text{and} \quad |a^2| < 1,\ \ k^2 < |a^2|$$

the series expansion is

$$\Pi(\varphi, a^2, k) = \sum_{m=0}^{\infty} b_m k^{2m}, \tag{29}$$

where

$$b_m = \binom{-\frac{1}{2}}{m}(-)^m \int_0^{\varphi} \frac{\sin^{2m}\theta\, d\theta}{1 - a^2\sin^2\theta}.$$

Hence

$$b_o = \begin{cases} \dfrac{1}{\sqrt{1-a^2}} \tan^{-1}[\sqrt{1-a^2}\ \tan\varphi] & a^2 < 1 \quad (30) \\[2ex] \dfrac{1}{2\sqrt{a^2-1}} \log \left| \dfrac{\sqrt{a^2-1}\ \tan\varphi + 1}{\sqrt{a^2-1}\ \tan\varphi - 1} \right| & a^2 > 1 \quad (31) \end{cases}$$

$$b_1 = \frac{b_o - \varphi}{2a^2}, \qquad b_2 = \frac{1}{16a^4}\,[3a^2 \sin\varphi\cos\varphi + 6\,b_o - 3(2+a^2)\varphi]\,;$$

and the recurrence relation is

$$2(m+1)a^2\, b_{m+1} = (2m+1+2ma^2)\,b_m + (1-2m)\,b_{m-1} - (-)^m \binom{-\frac{1}{2}}{m-1} \sin^{2m-1}\varphi\cos\varphi$$
$$- (-)^m \binom{\frac{1}{2}}{m} \int_o^{\varphi} \sin^{2m}\theta\, d\theta\,.$$

In the region $|a^2| < 1, \quad k^2 < 1$

a different series expansion is valid,

$$\Pi(\varphi, a^2, k) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} (a^2)^m \binom{-\frac{1}{2}}{j} \left(\frac{k^2}{-a^2}\right)^j \int_o^{\varphi} \sin^{2m}\theta\, d\theta\,. \qquad (32)$$

Both series can be derived very easily. The series in Eq. (29) is obtained by expanding the quantity

$$(1 - k^2 \sin^2\theta)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-k^2)^m \sin^{2m}\theta$$

and the series in Eq. (32), by the above expansion times the geometric series expansion of

$$(1 - a^2 \sin^2\theta)^{-1} = \sum_{j=0}^{\infty} (a^2 \sin^2\theta)^j.$$

Term by term integrations are then carried out for both expansions.

The regions of validity of the two series, with their overlapping region $1 > |a^2| > k^2$ are shown in Fig. 2.

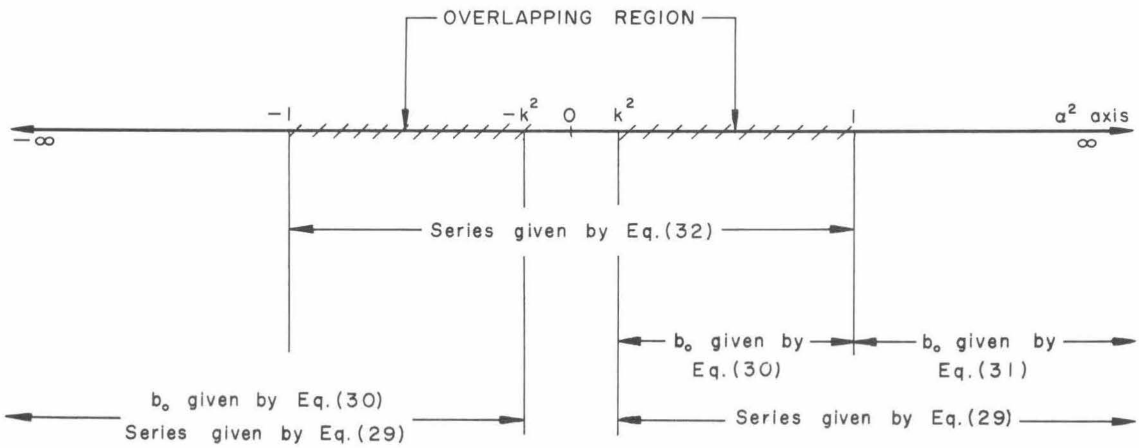

Figure 2

We have chosen to evaluate the integral in this common region by Eq. (29). For values of φ and k such that $k^2 \sin^2 \varphi$ is near one, both series expansions for $\Pi(\varphi, a^2, k)$ converge slowly and hence certain addition formulae [5]* must be used to facilitate these calculations. These addition formulae are of the form

$$\Pi(\varphi, a^2, k) \pm \Pi(\beta, a^2, k) = \Pi(\theta, a^2, k) \mp Q \tag{33}$$

where

$$\theta = 2 \tan^{-1}\left[\frac{\sin\varphi \sqrt{1-k^2 \sin^2 \beta} \pm \sin\beta \sqrt{1-k^2 \sin^2 \varphi}}{\cos\varphi + \cos\beta}\right] \tag{34}$$

or

$$\theta = \cos^{-1}\left[\frac{\cos\varphi \cos\beta \mp \sin\varphi \sin\beta \sqrt{(1-k^2 \sin^2 \varphi)(1-k^2 \sin^2 \beta)}}{1-k^2 \sin^2 \varphi \sin^2 \beta}\right] . \tag{35}$$

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* Some errors of sign for these formulae were found in this reference.

We now let

$$Q_a = \sqrt{\frac{a^2}{(1-a^2)(a^2-k^2)}}\ \tan^{-1}\left[\frac{\sin\varphi\ \sin\beta\ \sin\theta\ \sqrt{a^2(1-a^2)(a^2-k^2)}}{1-a^2\ \sin^2\theta \pm a^2\ \sin\varphi\ \sin\beta\ \cos\theta\ \sqrt{1-k^2\sin^2\theta}}\right] \tag{36}$$

and

$$Q_b = \sqrt{\frac{a^2}{(a^2-1)(a^2-k^2)}}\ \tanh^{-1}\left[\frac{\sin\varphi\ \sin\beta\ \sin\theta\ \sqrt{a^2(a^2-1)(a^2-k^2)}}{1-a^2\sin^2\theta \pm a^2\sin\varphi\ \sin\beta\ \cos\theta\ \sqrt{1-k^2\ \sin^2\theta}}\right]. \tag{37}$$

Then for

$$\begin{aligned} -\infty < a^2 < 0 \quad &, \quad Q = -Q_a; \\ 0 < a^2 < k^2 \quad &, \quad Q = Q_b; \\ k^2 < a^2 < 1 \quad &, \quad Q = Q_a; \\ 1 < a^2 < \infty \quad &, \quad Q = Q_b \quad . \end{aligned} \tag{38}$$

In order to apply these formulae to our problem one must assign a value to β, choose a sign and then calculate a θ from the given φ and k. The proper choice of β and sign will produce a $\theta < \varphi$ and hence the series for $\Pi(\theta, a^2, k)$ will converge with fewer terms than the series for $\Pi(\varphi, a^2, k)$.

Obviously the choice of β and the sign is rather important. The minimum θ occurs when $\beta = \pi/4$ and the lower sign is chosen. Making this substitution and choosing Eq. (34) the more convenient form for θ, the addition formulae can be written as

$$\Pi(\varphi, a^2, k) = \Pi(\theta, a^2, k) + \Pi(\pi/4, a^2, k) + Q \tag{39}$$

where

$$\theta = 2 \tan^{-1}\left[\frac{\sin\varphi\sqrt{1-\frac{k^2}{2}} - \sqrt{\frac{1}{2}\left(1-k^2\sin^2\varphi\right)}}{\cos\varphi + \sqrt{\frac{1}{2}}}\right] \tag{40}$$

and now

$$Q_a = \sqrt{\frac{a^2}{(1-a^2)(a^2-k^2)}}\tan^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta\sqrt{a^2(1-a^2)(a^2-k^2)}}{1-a^2\sin^2\theta-\frac{a^2}{\sqrt{2}}\sin\varphi\cos\theta\sqrt{1-k^2\sin^2\theta}}\right], \tag{41}$$

$$Q_b = \sqrt{\frac{a^2}{(a^2-1)(a^2-k^2)}}\tanh^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta\sqrt{a^2(a^2-1)(a^2-k^2)}}{1-a^2\sin^2\theta-\frac{a^2}{\sqrt{2}}\sin\varphi\cos\theta\sqrt{1-k^2\sin^2\theta}}\right], \tag{42}$$

and Q is given by Eq. (38).

This reduction in θ is illustrated in Fig. 3 where θ/φ is plotted as a function of φ for various values of k^2. At first glance one might decide that the addition formula should be applied whenever $\varphi > \pi/4$. It must be remembered, however, that whenever the addition formula is used the series for $\Pi(\pi/4, a^2, k)$ must be evaluated as well as the one for $\Pi(\theta, a^2, k)$ and hence the combined number of terms required must be compared against the number of terms required for the single series expansion of $\Pi(\varphi, a^2, k)$.

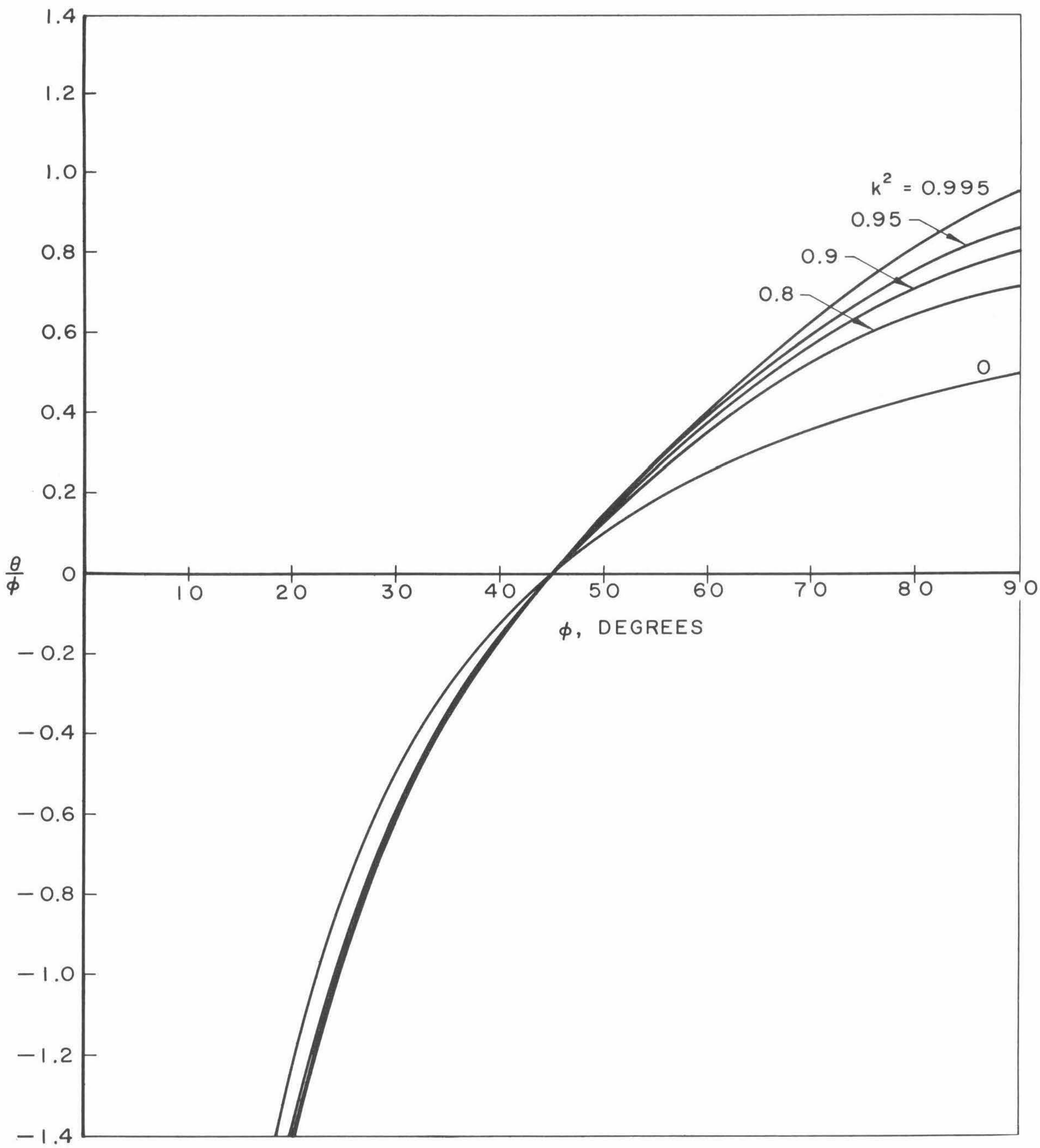

Figure 3. Variation of θ/φ with φ for several values of the modulus k^2. θ is calculated by Eq. (34).

For values of k^2 very near one, the reduction in φ is not sufficient to assure fast convergence, and therefore repeated application of the addition formula can be carried out. Let θ_n be the final value of θ after the addition formula has been applied n times. Then the integral can be expressed as follows:

$$\Pi(\varphi, a^2, k) = n\Pi(\pi/4, a^2, k) + \Pi(\theta_n, a^2, k) + Q \tag{43}$$

where

$$\theta_1 = 2\tan^{-1}\left[\frac{\sin\varphi\sqrt{1-\frac{k^2}{2}} - \sqrt{\frac{1}{2}(1-k^2\sin^2\varphi)}}{\cos\varphi + \sqrt{\frac{1}{2}}}\right],$$

$$\theta_2 = 2\tan^{-1}\left[\frac{\sin\theta_1\sqrt{1-\frac{k^2}{2}} - \sqrt{\frac{1}{2}(1-k^2\sin^2\theta_1)}}{\cos\theta_1 + \sqrt{\frac{1}{2}}}\right], \tag{44}$$

$$\vdots$$

$$\theta_n = 2\tan^{-1}\left[\frac{\sin\theta_{n-1}\sqrt{1-\frac{k^2}{2}} - \sqrt{\frac{1}{2}(1-k^2\sin^2\theta_{n-1})}}{\cos\theta_{n-1} + \sqrt{\frac{1}{2}}}\right].$$

so that,

$$Q_a = \sqrt{\frac{a^2}{(1-a^2)(a^2-k^2)}}\left\{\tan^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta_1\sqrt{a^2(1-a^2)(a^2-k^2)}}{1-a^2(\sin^2\theta_1+\sqrt{\frac{1}{2}}\sin\varphi\cos\theta_1\sqrt{1-k^2\sin^2\theta_1})}\right]\right.$$

$$+ \;.\;.\;.\;.\; + \;.\;.\;.\;.$$

$$\left. + \tan^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta_n\sqrt{a^2(1-a^2)(a^2-k^2)}}{1-a^2(\sin^2\theta_n+\sqrt{\frac{1}{2}}\sin\varphi\cos\theta_n\sqrt{1-k^2\sin^2\theta_n})}\right]\right\}, \tag{45}$$

and

$$Q_b = \sqrt{\frac{a^2}{(a^2-1)(a^2-k^2)}}\left\{\tanh^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta_1\sqrt{a^2(a^2-1)(a^2-k^2)}}{1-a^2(\sin^2\theta_1+\sqrt{\frac{1}{2}}\sin\varphi\cos\theta_1\sqrt{1-k^2\sin^2\theta_1})}\right]\right.$$

$$+\;.\;.\;.\;.\;.\;+\;.\;.\;.\;.\;.$$

$$\left.+\tanh^{-1}\left[\frac{\sqrt{\frac{1}{2}}\sin\varphi\sin\theta_n\sqrt{a^2(a^2-1)(a^2-k^2)}}{1-a^2(\sin^2\theta_n+\sqrt{\frac{1}{2}}\sin\varphi\cos\theta_n\sqrt{1-k^2\sin^2\theta_n})}\right]\right\}. \quad (46)$$

Again Q is given by Eq. (38).

Table I illustrates how well repeated application of the addition formula works for the most difficult cases expected to be encountered. In this table values of θ_n for $n = 1, 2, 3, 4, 5$ are shown for $\varphi = \pi/2$ and $k^2 = 0.9990$ and 0.9999.

Table I Reduction of the Argument

k^2	φ	θ_1	θ_2	θ_3	θ_4	θ_5
0.9990	$\pi/2$	1.5392	1.4820	1.3518	1.0499	0.4271
0.9999	$\pi/2$	1.5608	1.5425	1.5008	1.4015	1.1664

Test runs were made with our program and the total number of terms were recorded for cases when the addition formula was used and when it was not used. We found that the least computer time resulted if we used the addition formula for values of φ and k such that

$k \sin \varphi > 0.75$. Furthermore, the addition formula is repeatedly used until $k \sin \theta_n < 0.75$. In this way we were able to calculate $\Pi(\varphi, \alpha^2, k)$ for all ranges of α^2, $|\alpha^2 - k^2| > 0.1$,* where $0 < \varphi < 1.570$ and $0 < k^2 < 0.999$ using less than a total of 25 terms.

The flow chart and a copy of the subroutine are given in Appendix C.

* The details of the region $|\alpha^2 - k^2| \lesssim 0.1$ are given in Appendix C.

APPENDICES

Appendix A. Approximate functions for the complete elliptic integrals of the first and second kind.

The complete elliptic integrals of the first and second kind, i. e. ,

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{A.1}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \quad d\theta \tag{A.2}$$

where $0 \leqslant k^2 < 1$, can be calculated by the approximate functions [3]

$$\begin{aligned} K^*(k) = & \{a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4 \} \\ & + \{b_0 + b_1 \eta + b_2 \eta^2 + b_3 \eta^3 + b_4 \eta^4 \} \log \frac{1}{\eta} \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} E^*(k) = & \{1 + c_1 \eta + c_2 \eta^2 + c_3 \eta^3 + c_4 \eta^4 \} \\ & + \{d_1 \eta + d_2 \eta^2 + d_3 \eta^3 + d_4 \eta^4 \} \log \frac{1}{\eta} \end{aligned} \tag{A.4}$$

In these equations

$$\eta = 1 - k^2 = k'^2$$

and the coefficients as given in Ref. [3] are

$a_o = 1.3862,9436,112$ $b_o = 0.5$

$a_1 = 0.0966,6344,259$ $b_1 = 0.1249,8593,597$

$a_2 = 0.0359,0092,383$ $b_2 = 0.0688,0248,576$

$a_3 = 0.0374,2563,713$ $b_3 = 0.0332,8355,346$

$a_4 = 0.0145,1196,212$ $b_4 = 0.0044,1787,012$

$c_1 = 0.4432,5141,463$ $d_1 = 0.2499,8368,310$

$c_2 = 0.0626,0601,220$ $d_2 = 0.0920,0180,037$

$c_3 = 0.0475,7383,546$ $d_3 = 0.0406,9697,526$

$c_4 = 0.0173,6506,451$ $d_4 = 0.0052,6449,639$

The error curves of these approximation functions are shown in Fig. A-1.

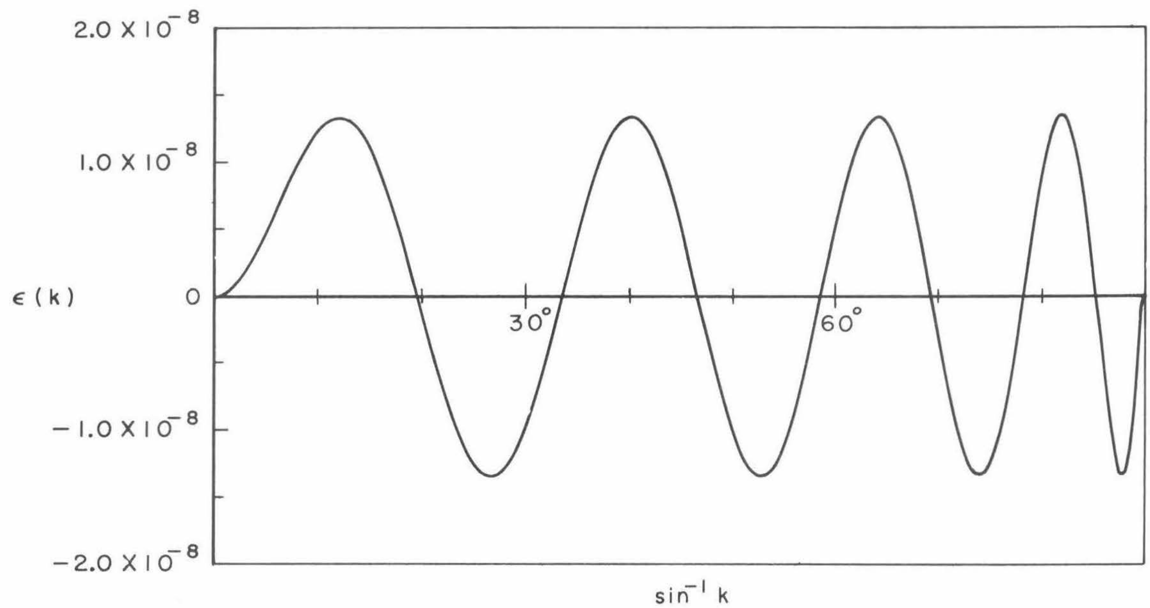

Figure A-1. Error in K^* and E^* as a function of $\sin^{-1}k$. This figure is given in Ref. [3], p 172.

The subroutines of these functions are not included since they are quite straightforward.

Appendix B: The Evaluation of the Incomplete Elliptic Integrals of the First and Second Kind.

The incomplete elliptic integrals of the first and second kind

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{B.1a}$$

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} \quad d\theta \tag{B.1b}$$

are evaluated by two types of series and some special formulae for the boundary condition cases. The reader may refer to Ref. 4 for detailed discussion of these series expansions.

In the range $0 < |k \sin\varphi| < \tanh 1 = 0.7615,942$ the series expansions are evaluated by the equations

$$F(\varphi, k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} k^{2n} \int_0^{\varphi} \sin^{2n} \theta \, d\theta \tag{B.2a}$$

and

$$E(\varphi, k) = -\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2 (2n-1)} k^{2n} \int_0^{\varphi} \sin^{2n} \theta \, d\theta \tag{B.2b}$$

These formulae can be derived directly by the binomial expansion of the integrands of $F(\varphi, k)$ and $E(\varphi, k)$ in Eqs. (B.1). Though these series are convergent for the entire range of φ and k, their convergence is slow when $|k\sin\varphi| > \tanh 1$. A second set of series therefore is used to evaluate $F(\varphi,k)$ and $E(\varphi,k)$ in the range

$$\tanh 1 < |k\sin\varphi| < 1 .$$

These are presented in Ref. 4 and are

$$F(\varphi,k) = \frac{2}{\pi} K' \log\left[\frac{4}{\sqrt{1-k^2x^2} + |k|\sqrt{1-x^2}}\right] + |k|\sqrt{\frac{1-x^2}{1-k^2x^2}} \log\left[\frac{1+|kx|}{2}\right]$$
$$- |k|\sqrt{(1-x^2)(1-k^2x^2)} \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n+1)!} (1-k^2x^2)^n \sum_{m=n+1}^{\infty} \left[\frac{(2m+2)!}{2^{2m+2}(m+1)!^2}\right]^2 (k')^{2m-2n}$$
$$- \sum_{n=1}^{\infty} \left[\frac{(2n)!}{2^{2n} n!^2}\right]^2 (k')^{2n} \sum_{m=1}^{n} \frac{1}{m(2m-1)} , \qquad \text{(B.3a)}$$

$$E(\varphi,k) = \frac{2}{\pi} (K' - E') \log\left[\frac{4}{\sqrt{1-k^2x^2} + |k|\sqrt{1-x^2}}\right] - |k|\sqrt{\frac{1-x^2}{1-k^2x^2}} (1 - |kx|)$$
$$- |k|\sqrt{(1-x^2)(1-k^2x^2)} \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n+1)!} (1-k^2x^2)^n \sum_{m=n+1}^{\infty} \left[\frac{(2m)!}{2^{2m} m!^2}\right]^2 \frac{2m+1}{2m+2} (k')^{2m-2n}$$
$$- \sum_{n=1}^{\infty} \left[\frac{(2n)!}{2^{2n} n!^2}\right]^2 \frac{2n}{(2n-1)} (k')^{2n} \sum_{m=1}^{n} \frac{1}{m(2m-1)} + \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n} n!^2}\right]^2 \frac{1}{(2n-1)^2} (k')^{2n} \qquad \text{(B.3b)}$$

where

$$x = \sin\varphi .$$

Our subroutine for $F(\varphi, k)$ and $E(\varphi, k)$ follows closely the one suggested in Ref. 4, pages 11 to 14. The flow chart and a copy of the subroutine written in Fortran IV language for the IBM 7090 are given in Appendix C.

The maximum number of terms required to yield six decimal places in the neighborhood of $k \sin \varphi = \tanh 1$ is nineteen terms for the first series method (Eqs. (B.2)), and fourteen terms for the second method (Eqs. (B.3)). It should also be pointed out that even though Eqs. (B.3) are theoretically good in the regions of $k \sin \varphi = 1$, when $k^2 \sin^2 \varphi$ exceeds 0.99 the number of significant figures is reduced. This loss in accuracy occurs because the quantity $1 - k^2 \sin^2 \varphi$ must be evaluated.

A few words are also needed for the boundary condition cases:

Case 1: k^2 near 0

This presents no special problem since the first series (B.2) will converge rapidly. We have arbitrarily considered $k^2 < 10^{-7}$ to be equal to zero and hence only the first terms of the series in Eqs. (B.2) are required yielding

$$F(\varphi, 0) = \varphi$$

and

$$E(\varphi, 0) = \varphi \quad .$$

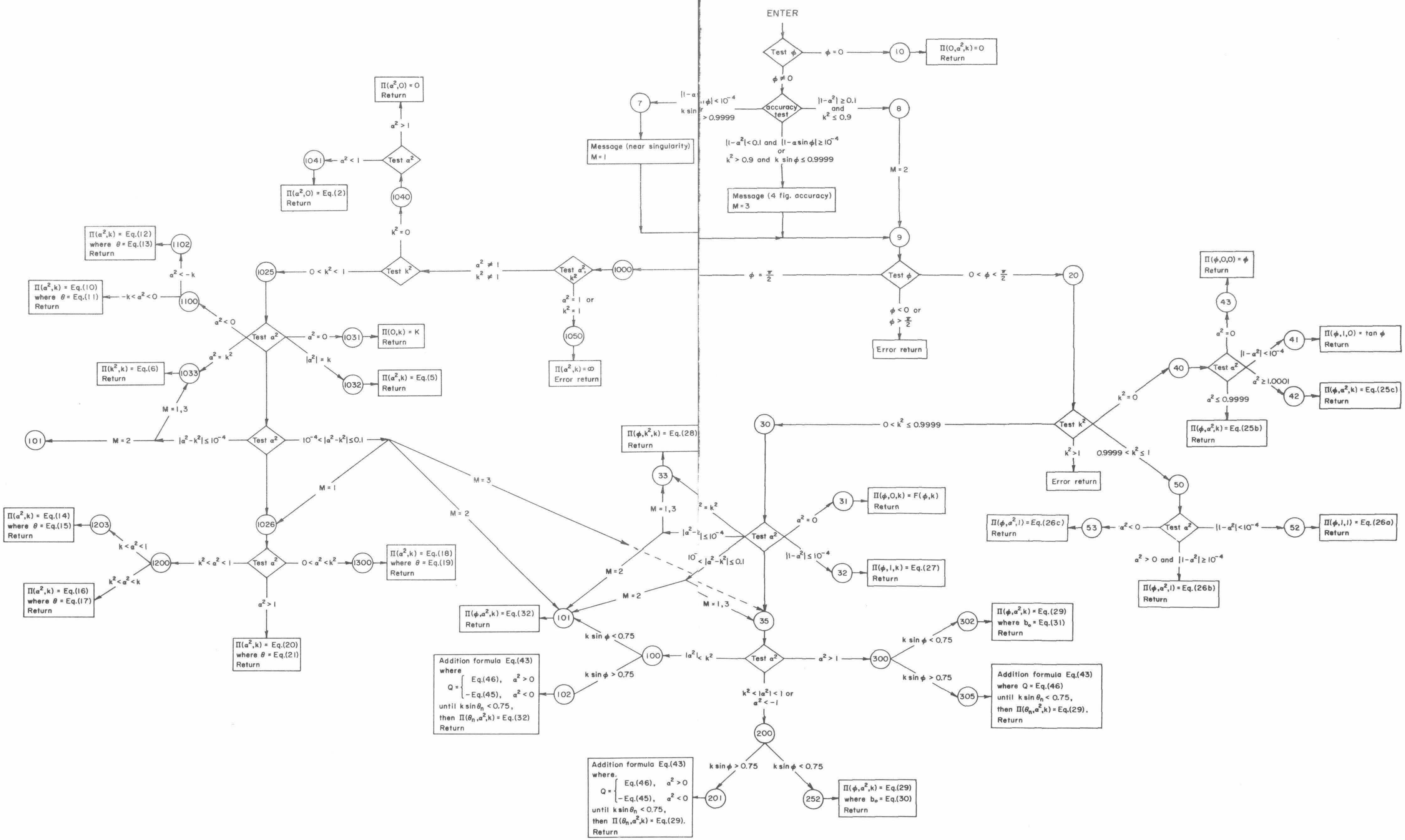

Figure C-1. The flow chart of Subroutine PIX for calculation of the complete or incomplete elliptic integral of the third kind. The circled numbers in the flow chart refer to statement numbers in the program listing in Fig. C-2.

Case 2: $k^2 \sim 1$ or $k'^2 = 1 - k^2 \sim 0$

$F(\varphi, k)$ and $E(\varphi, k)$ have the following series expansions [5] in powers of k'^2

$$F(\varphi, k) = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} k'^{2m} \rho_{2m}(\varphi) \qquad [0 < k'^2 \tan^2 \varphi < 1, k^2 < 1] \tag{B.4a}$$

where

$$\rho_0(\varphi) = \log \frac{1+\sin\varphi}{\cos\varphi}$$

$$\rho_2(\varphi) = \frac{\sin\varphi \sec^2\varphi}{2} - \tfrac{1}{2} \log \frac{1+\sin\varphi}{\cos\varphi}$$

$$\rho_4(\varphi) = \tfrac{1}{8}[2 \sin^3\varphi \sec^4\varphi - 3 \sin\varphi \sec^2\varphi + 3 \log \frac{1+\sin\varphi}{\cos\varphi}]$$

.

.

$$\rho_{2m}(\varphi) = \frac{1}{2m}[\sin^{2m-1}\varphi \sec^{2m}\varphi + (1-2m)\rho_{2m-2}(\varphi)]$$

and

$$E(\varphi, k) = \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} k'^{2m} d_{2m}(\varphi) \qquad [0 < k'^2 \tan^2\varphi < 1; k^2 < 1] \tag{B.4b}$$

where

$$d_0(\varphi) = \sin\varphi$$

$$d_2(\varphi) = -\sin\varphi + \log \frac{1+\sin\varphi}{\cos\varphi}$$

$$d_4(\varphi) = \tfrac{1}{2}[\sin^3\varphi \sec^2\varphi + 3\sin\varphi - 3 \log \frac{1+\sin\varphi}{\cos\varphi}]$$

.

.

$$d_{2m}(\varphi) = \frac{1}{2(m-1)}[\sin^{2m-1}\varphi \sec^{2(m-1)}\varphi + (1-2m)d_{2(m-1)}(\varphi)]$$

with $m \neq 1$.

Again we have arbitrarily considered 10^{-7} our criterion for zero. That is we have taken $|k'^2| \leq 10^{-7}$ to be $k'^2 = 0$ or $|1.0 - k^2| \leq 10^{-7}$ to be the same as $k^2 = 1.0$ so that F and E are represented by their first terms in Eqs. (B.4) giving

$$F(\varphi, 1) = \log (\tan\varphi + \sec\varphi)$$

and

$$E(\varphi, 1) = \sin\varphi \ .$$

The flow chart and Fortran IV program are given in Appendix C.

Appendix C. The most frequently used symbols in the subroutine listings in Figs. C-2 and C-4 have the following definitions:

AS		α^2
XK		k
SK		k^2
PK		k'
PHI		φ
THE		θ
K		K^*
Ec		E^*
F		$F(\varphi, k)$
E		$E(\varphi, k)$
PI		$\Pi(\varphi, \alpha^2, k)$ or $\Pi(\alpha^2, k)$
TOL		Allowable error
M		Accuracy or error indicator - (output)
		M = 1, less than four figure accuracy or error return due to non-convergence, overflow or illegal parameter in argument
		M = 2, six figure accuracy
		M = 3, four to six figure accuracy
I	a)	Indicator in subroutine PIX - (output)
		I = 0, θ, F, and E were not calculated
		I = 1, θ, $F(\theta, k')$ and $E(\theta, k')$ were calculated
		I = 2, θ, $F(\theta, k)$ and $E(\theta, k)$ were calculated

b) Indicator in subroutine ELLI - (input)

I = 1, both F and E to be calculated

I = 2, calculate E only

I = 3, calculate F only

Figures C-1 and C-2 are the flow chart and Fortran IV listing for the subroutine which calculates $\Pi(\alpha^2, k)$ or $\Pi(\varphi, \alpha^2, k)$ the complete or incomplete elliptic integral of the third kind. The circled numbers in Fig. C-1 refer to statement numbers in the program listing given in Fig. C-2.

Six figure accuracy is guaranteed except when $|1-\alpha^2| < 0.1$ or $k^2 > 0.9$ and at least four figure accuracy is guaranteed when $|1-\alpha \sin\varphi| \geqslant 0.0001$ and $k \sin\varphi \leqslant 0.9999$. The accuracy indicator M is set as described above and a message is printed whenever the accuracy is less than six figures.

If $k \sin\varphi$ is greater than 0.75 or $\varphi = \pi/2$ the methods outlined in the text cannot calculate $\Pi(\alpha^2, k)$ or $\Pi(\varphi, \alpha^2, k)$ to six figures when $|\alpha^2 - k^2| \leqslant 0.1$ or to four figures when $|\alpha^2 - k^2| \leqslant 0.0001$, because accuracy is lost in calculating the quantity $\alpha^2 - k^2$. Therefore, our program transfers control to statement number 101 using Eq. (32), without the addition formulae for both integrals whenever $|\alpha^2 - k^2|$ is less than or equal to 0.1 and M = 2 (six figure accuracy); thus the quantity $\alpha^2 - k^2$ need not be evaluated. When $|\alpha^2 - k^2| \leqslant 0.0001$ and no more than four figure accuracy is required (M = 1 or M = 3), α^2 is considered to be the same as k^2 so that

$$\Pi(\varphi, \alpha^2, k) = \Pi(\varphi, k^2, k)$$

or

$$\Pi(\alpha^2, k) = \Pi(k^2, k).$$

The remaining deviation from the text is the case when $\varphi = \pi/2$, $M = 3$ and $0.0001 < |\alpha^2 - k^2| < 0.1$. For this case better accuracy is obtained if $\Pi(\alpha^2, k)$ is calculated as a special case of the incomplete integral.

The flow chart and Fortran IV listing for the subroutine which calculates the incomplete elliptic integrals of the first and second kind are given in Figs. C-3 and C-4 respectively. As above, the circled numbers in the flow chart correspond to statement numbers in the program listing.

Figure C-2. The Fortran IV listing of Subroutine PIX.

```
      SUBROUTINE PIX(PHI,AS,XK,TOL,M,PI,K,EC,I,THE,F,E)
    1 FORMAT(19H PHI NEG OR GT PI/2)
    2 FORMAT(7H K GT 1)
    3 FORMAT(30H PI AT LEAST 4 FIGURE ACCURACY)
    4 FORMAT(15H OVERFLOW IN PI)
    5 FORMAT(17H PI NOT CONVERGED)
    6 FORMAT(20H PI NEAR SINGULARITY)
 1007 FORMAT(22H AS OR SK=ONE, PIC=INF)
      REAL K,N
      DIMENSION B(101)
      TAL=0.
      PIT=0.
      Q=0.
      NN=1
      THE=0.
      PI=0.
      N=0.
      I=0
      EC=0.
      K=0.
      F=0.
      E=0.
      CALL OVERFL(L)
      IF(ABS(PHI).LE..0000001) GO TO 10
      SK=XK*XK
      SP=SIN(PHI)
      A=SQRT(ABS(AS))
      ASP=A*SP
      IF(AS.LT.0.) ASP=-ASP
      IF(ABS(1.0-AS).GE..1.AND.SK.LE..9) GO TO 8
      IF(ABS(1.0-ASP).LT..0001.OR.XK*SP.GT..9999) GO TO 7
      WRITE(6,3)
      M=3
      GO TO 9
    7 WRITE(6,6)
      M=1
      GO TO 9
    8 M=2
    9 IF(.0000001.LT.PHI.AND.PHI.LT.1.5707962) GO TO 20
      IF(ABS(PHI-1.5707963).LE..0000001) GO TO 1000
      WRITE(6,1)
      M=1
      RETURN
   10 PI=0.
      M=2
      RETURN
   20 CP=COS(PHI)
      SPS=SP*SP
      IF(.0000001.LT.SK.AND.SK.LE..9999) GO TO 30
      IF(SK.LE..0000001) GO TO 40
      IF(SK.LT.1.0000001) GO TO 50
      WRITE (6,2)
      M=1
      RETURN
   40 IF(ABS(AS).LE..0000001) GO TO 43
      IF(ABS(AS-1.0).LT..0001) GO TO 41
      IF(AS.GT.1.0) GO TO 42
      X=SQRT(1.0-AS)
      PI=ATAN(X*SP/CP)/X
      RETURN
   41 PI=SP/CP
      RETURN
   42 X=SQRT(AS-1.0)
      PI=.5/X*ALOG(ABS(((X*SP/CP)+1.)/((X*SP/CP)-1.0)))
      RETURN
   43 PI=PHI
      M=2
      RETURN
   50 IF(ABS(AS-1.0).LT..0001) GO TO 52
      IF(AS.LT.0.) GO TO 53
      PI=(ALOG((SP+1.0)/CP)-0.5*A*ALOG(ABS((1.0+ASP)/(1.0-ASP))))/
     1 (1.0-AS)
      RETURN
   52 PI=.5*(SP/(CP*CP)+.5*ALOG((1.0+SP)/(1.0-SP)))
      RETURN
   53 PI=(ALOG((SP+1.0)/CP)+A*ATAN(-ASP))/(1.0-AS)
      RETURN
   30 IF(ABS(AS).LE..0000001) GO TO 31
      IF(ABS(AS-1.0).LT..0001) GO TO 32
      IF(ABS(AS-SK).LE..0000001) GO TU 33
      IF(ABS(AS-SK).LE..0001) GO TO (33,101,33),M
      IF(ABS(AS-SK).LE.0.1) GO TO (35,101,35),M
      GO TO 35
   31 CALL ELLI(PHI,XK,F,E,3,L,TOL)
      PI=F
      RETURN
   32 CALL ELLI(PHI,XK,F,E,1,L,TOL)
      PI=((1.0-SK)*F-E+SP/CP*SQRT(1.0-SK*SPS))/(1.0-SK)
      RETURN
   33 CALL ELLI(PHI,XK,F,E,2,L,TOL)
      PI=(E-SK*SP*CP/SQRT(1.0-SK*SPS))/(1.0-SK)
      RETURN
   35 IF(ABS(AS).LT.SK) GO TO 100
      IF(SK.LT.ABS(AS).AND.ABS(AS).LT.1.0) GO TO 200
      IF(AS.GT.1.0) GO TO 300
  200 IF(XK*SP.LT..75) GO TO 252
  201 THE=2.0*ATAN((SQRT(1.0-.5*SK)*SP-.70710678*SQRT(1.0-SK*SPS))/
     1 (CP+.70710678))
      ST=SIN(THE)
      CT=COS(THE)
      STS=ST*ST
      Q=SQRT(AS/((1.0-AS)*(AS-SK)))*ATAN(+.70710678*SP*ST*SQRT(AS*(1.0-
     1 AS)*(AS-SK))/(1.0-AS*STS-.70710678*AS*SP*CT*SQRT(1.-SK*STS)))+Q
      PHI=THE
      SP=ST
      CP=CT
      SPS=SP*SP
      TAL=TAL+1.0
      IF(XK*SP.GT..75) GO TO 201
      NN=2
      GO TO 252
  210 PIT=PI
      T1=N
      NN=3
      PHI=.78539816
      SP=.70710678
      CP=.70710678
      SPS=.5
      GO TO 252
  211 IF(AS.LT.0.) GO TO 212
      PI=TAL*PI+PIT+Q
      RETURN
  212 PI=TAL*PI+PIT-Q
      RETURN
  252 SAS=SQRT(1.0-AS)
      BO=ATAN(SAS*SP/CP)/SAS
  251 B(1)=.5*(BO-PHI)/AS
      SC=SP*CP
      TK=SK*SK
      B(2)=.0625*(3.0*AS*SC+6.0*BO-(6.0+3.0*AS)*PHI)/(AS*AS)
      PI=BO+B(1)*SK+B(2)*TK
      DM=-SC
      EM=-.5
      T=.5*(PHI-SC)
      DO 260 J=2,100
      N=J
      TN=2*J
      SC=SC*SPS
      DM=DM*(N-1.5)*SPS/(N-1.0)
      EM=EM*(N-1.5)/N
      T=((TN-1.0)*T-SC)/TN
      B(J+1)=((TN+1.0+TN*AS)*B(J)+(1.0-TN)*B(J-1)-DM-EM*T)/(AS*(TN+2.0))
      TK=SK*TK
      TERM=B(J+1)*TK
      PI=PI+TERM
      IF(ABS(TERM).LE.TOL) GO TO (501,210,211,310,311),NN
  260 CONTINUE
  261 WRITE(6,5)
      M=1
      RETURN
  100 IF(XK*SP.GT..75) GO TO 102
  101 SC=SP*CP
      AK=-SK/AS
      T=.5*(PHI-SC)
      AT=AS
```

Figure C-2 continued

```
      C=-.5*AK
      CM=1.0+C
      PI=CM*AT*T
      DO 110 MM=2,200
      N=MM
      TN=2*MM
      SC=SC*SPS
      T=((TN-1.0)*T-SC)/TN
      AT=AT*AS
      C=C*(0.5-N)*AK/N
      CM=CM+C
      SM=CM*AT*T
      PI=PI+SM
      IF(ABS(SM/(PI+PHI)).LE.TOL) GO TO 103
      IF(ABS(AT).GT.1.0 E-30) GO TO 110
      AT=AT*1.0 E30
      C=C*1.0 E-30
      CM=CM*1.0 E-30
  110 CONTINUE
      GO TO 261
  103 PI=PI+PHI
      GO TO (501,121,122),NN
  102 THE=2.0*ATAN((SQRT(1.0-.5*SK)*SP-.70710678*SQRT(1.0-SK*SPS))/
     1 (CP+.70710678))
      ST=SIN(THE)
      CT=COS(THE)
      STS=ST*ST
      IF(AS.LT.0.) GO TO 105
      Y =+.70710678*SP*ST*SQRT(AS*(AS-1.0)*(AS-SK))/
     1  (1.0-AS*STS-.70710678*SP*CT*SQRT(1.0-SK*STS)*AS)
      TY=.5*ALOG(ABS((1.0+Y)/(1.0-Y)))
      Q=SQRT(AS/((AS-1.0)*(AS-SK)))*TY+Q
  104 PHI=THE
      SP=ST
      CP=CT
      SPS=STS
      TAL=TAL+1.0
      IF(XK*SP.GT..75) GO TO 102
      NN=2
      GO TO 101
  105 Q=-SQRT(AS/((1.0-AS)*(AS-SK)))*ATAN(.70710678*SP*ST*SQRT(AS*(1.0-
     1 AS)*(AS-SK))/(1.0-AS*STS-.70710678*AS*SP*CT*SQRT(1.-SK*STS)))+Q
      GO TO 104
  121 PIT=PI
      NN=3
      PHI=.78539816
      SP=.70710678
      CP=.70710678
      SPS=.5
      GO TO 101
  122 PI=TAL*PI+PIT+Q
      RETURN
  300 IF(XK*SP.GT..75) GO TO 305
  302 SAD=SQRT(AS-1.0)
      XX=SAD*SP/CP
      BO=.5/SAD*ALOG(ABS((XX+1.0)/(XX-1.0)))
      GO TO 251
  305 THE=2.0*ATAN((SQRT(1.0-.5*SK)*SP-.70710678*SQRT(1.0-SK*SPS))/
     1 (CP+.70710678))
      ST=SIN(THE)
      CT=COS(THE)
      STS=ST*ST
      Y =+.70710678*SP*ST*SQRT(AS*(AS-1.0)*(AS-SK))/
     1  (1.0-AS*STS-.70710678*SP*CT*SQRT(1.0-SK*STS)*AS)
      TY=.5*ALOG(ABS((1.0+Y)/(1.0-Y)))
      Q=SQRT(AS/((AS-1.0)*(AS-SK)))*TY+Q
      PHI=THE
      SP=ST
      CP=CT
      SPS=STS
      TAL=TAL+1.0
      IF(XK*SP.GT..75) GO TO 305
      NN=4
      GO TO 302
  310 PIT=PI
      NN=5
      PHI=.78539816
      SP=.70710678
      CP=.70710678
      SPS=.5
      GO TO 302
  311 PI=TAL*PI+PIT+Q
  501 CALL OVERFL(L)
      IF(L.EQ.2) RETURN
      WRITE(6,4)
      M=1
      RETURN
 1000 IF(ABS(AS-1.0).LE..0000001.OR.ABS(SK-1.0).LE..0000001) GO TO 1050
      IF(.0000001.LT.SK.AND.SK.LT..99999990) GO TO 1025
      IF(SK.LE..0000001) GO TO 1040
 1006 WRITE(6,2)
      M=1
      RETURN
 1050 WRITE(6,1007)
      M=1
      RETURN
 1040 IF(AS.LT..99999990) GO TO 1041
      PI=0.
      M=2
      RETURN
 1041 PI= 1.5707963/SQRT(1.0-AS)
      RETURN
 1025 PK=SQRT(1.0-SK)
      CALL ELIPT(XK,K,EC,L)
      IF(ABS(AS).LE..0000001) GO TO 1031
      IF(ABS(ABS(AS)-XK).LE..0000001) GO TO 1032
      IF(AS.LT.0.) GO TO 1100
      IF(ABS(AS-SK).LE..0000001) GO TO 1033
      SPS=1.0
      CP=0.
      IF(ABS(AS-SK).LE..0001) GO TO (1033,101,1033),M
      IF(ABS(AS-SK).LE.0.1) GO TO (1026,101,35),M
 1026 IF(AS.GT.SK.AND.AS.LE..9999) GO TO 1200
      IF(AS.LT.SK.AND.AS.GT..0000001) GO TO 1300
      THE=ASIN(SQRT(1.0/AS))
      CALL ELLI(THE,XK,F,E,1,L,TOL)
      PI =(EC*F-K*E)*SQRT(AS/((AS-1.0)*(AS-SK)))
      I=2
      RETURN
 1031 PI=K
      RETURN
 1032 PI =0.25*(3.14159265+2.0*(1.0-AS)*K)/(1.0-AS)
      RETURN
 1033 PI= EC/(1.0-SK)
      RETURN
 1100 XXK=-XK
      IF(AS.LT.XXK) GO TO 1102
      THE=ASIN(SQRT(AS/(AS-SK)))
      CALL ELLI(THE,PK,F,E,1,L,TOL)
      PI= SK*K/(SK-AS)-AS*(EC*F+K*(E-F))/SQRT(AS*(1.0-AS)*(AS-SK))
      I=1
      RETURN
 1102 THE=ASIN(1.0/SQRT(1.0-AS))
      CALL ELLI(THE,PK,F,E,1,L,TOL)
      PI= K/(1.0-AS)+AS*(EC*F+K*(E-F)-1.5707963)/
     1 SQRT(AS*(1.0-AS)*(AS-SK))
      I=1
      RETURN
 1200 IF(AS.GT.XK) GO TO 1203
      THE=ASIN(SQRT((AS-SK)/(AS*(1.0-SK))))
      CALL ELLI(THE,PK,F,E,1,L,TOL)
      PI= (EC*F+K*(E-F))*SQRT(AS/((AS-SK)*(1.0-AS)))
      I=1
      RETURN
 1203 THE=ASIN(SQRT((1.0-AS)/(1.0-SK)))
      CALL ELLI(THE,PK,F,E,1,L,TOL)
      PI= K+(1.5707963-(EC*F+K*(E-F)))*SQRT(AS/((AS-SK)*(1.0-AS)))
      I=1
      RETURN
 1300 THE=ASIN(SQRT(AS/SK))
      CALL ELLI(THE,XK,F,E,1,L,TOL)
      PI= K+(K*E-EC*F)*SQRT(AS/((1.0-AS)*(SK-AS)))
      I=2
      RETURN
      END
```

For Figures C-3 and C-4 the captions should be interchanged.

Figure C-3. The flow chart of Subroutine ELLI for calculation of the incomplete elliptic integrals of the first and second kinds. The circled numbers in the flow chart refer to statement numbers in the program listing in Fig. C-4.

```
      SUBROUTINE ELLI(PHI,XK,F,E,I,M,TOL)
      REAL N
      CALL OVERFL(M)
      F=0.
      E=0.
      SK=XK*XK
      IF(.0000001.LT.SK.AND.SK.LT..99999990) GO TO 25
      IF(SK.LE..0000001) GO TO 20
      IF(ABS(XK-1.0).LE..0000001) GO TO 23
   19 WRITE(6,3)
    3 FORMAT(7H K GT 1)
      M=1
      RETURN
   20 E=PHI
      F=PHI
      M=2
      RETURN
   23 SP=SIN(PHI)
      E=SP
      M=2
      IF(I.EQ.2) RETURN
      F=ALOG(SP/(COS(PHI))+1.0/(COS(PHI)))
   10 CALL OVERFL(M)
      IF(M.EQ.1) WRITE(6,4)
    4 FORMAT(17H OVERFLOW IN ELLI)
      RETURN
   25 IF(.0000001.LT.PHI.AND.PHI.LT.1.5707960) GO TO 46
      IF(ABS(PHI).LE..0000001) GO TO 41
      IF(ABS(PHI-1.5707963).LE..0000003) GO TO 44
   40 WRITE (6,5)
    5 FORMAT(23H NEG PHI OR PHI GT PI/2)
      M=1
      RETURN
   41 E=0.
      F=0.
      M=2
      RETURN
   44 CALL ELIPT(XK,F,E,M)
      RETURN
   46 SP=SIN(PHI)
      AKP=XK*SP
      IF(AKP.GT..7615942) GO TO 45
   43 SK=XK*XK
      CE=-.5*SK
      CF=-CE
      SPS=SP*SP
      CP=COS(PHI)
      SCT=SP*CP
      T=.5*(PHI-SCT)
      E=CE*T
      F=CF*T
      DO 64 J=2,200
      N=J
      TN=2.0*N
      SCT=SCT*SPS
      T=((TN-1.0)*T-SCT)/TN
      A=-SK/N
      CF=CF*(.5-N)*A
      CE=CE*(1.5-N)*A
      TCE=CE*T
      E=E+TCE
      TCF=CF*T
      F=F+TCF
      GO TO (62,61,62),I
   62 IF(ABS(TCF).LE.TOL) GO TO 65
      GO TO 64
   61 IF(ABS(TCE).LE.TOL) GO TO 66
   64 CONTINUE
      M=1
      WRITE(6,6)
    6 FORMAT(19H ELLB NOT CONVERGED)
      RETURN
   65 F=PHI+F
   66 E=PHI+E
      GO TO 10
   45 A=XK*COS(PHI)
      B2=1.0-AKP*AKP
      IF(B2.GE..05) GO TO 48
      IF(B2.LT..0005) GO TO 47
      WRITE(6,1)
    1 FORMAT(32H F AND E AT LEAST 4 FIG.ACCURACY)
      M=3
      GO TO 49
   47 WRITE(6,2)
    2 FORMAT(34H F AND E LESS THAN 4 FIG. ACCURACY)
      M=1
      GO TO 49
   48 M=2
   49 B=SQRT(B2)
      ABL=ALOG(4.0/(A+B))
      ADB=A/B
      PKS=1.0-XK*XK
      XI=.5
      XJ=.25*PKS
      XL=1.0
      AB=A*B
      XM=-AB*PKS*.140625
      XN=-AB*PKS*.1875
      S1=XM-XJ*XL
      S2=XN-.25*PKS
      S3=XJ
      S4=.75*XJ
      DO 200 J=2,100
      N=J
      TN=2*J
      C1=TN-1.0
      C2=C1/TN
      C3=(TN+1.0)/(TN+2.0)
      C4=C3*PKS
      AB=AB*B2
      CN=AB*XI/TN
      CD2A=C2*PKS*XJ
      CD2B=PKS*XJ/(TN*TN)
      XI=C2*XI
      XJ=C3*C4*XJ
      XL=XL+1.0/(N*C1)
      XM=(XM-AB*XI/TN)*C3*C4
      XN=(XN-CN)*C4*C2
      D1=XM-XJ*XL
      D2=XN-CD2A*XL+CD2B
      S1=S1+D1
      S2=S2+D2
      S3=S3+XJ
      S4=S4+C3*XJ
      GO TO (90,91,90) ,I
   90 IF(ABS(D1).LE.TOL) GO TO 210
      GO TO 200
   91 IF(ABS(D2).LE.TOL) GO TO 211
  200 CONTINUE
      M=1
      WRITE (6,6)
      RETURN
  210 F=(1.0+S3)*ABL+ADB*ALOG(.5+.5*AKP)+S1
      IF(I.EQ.3) RETURN
  211 E=(.5+S4)*PKS*ABL+1.0-ADB*(1.0-AKP)+S2
      RETURN
      END
```

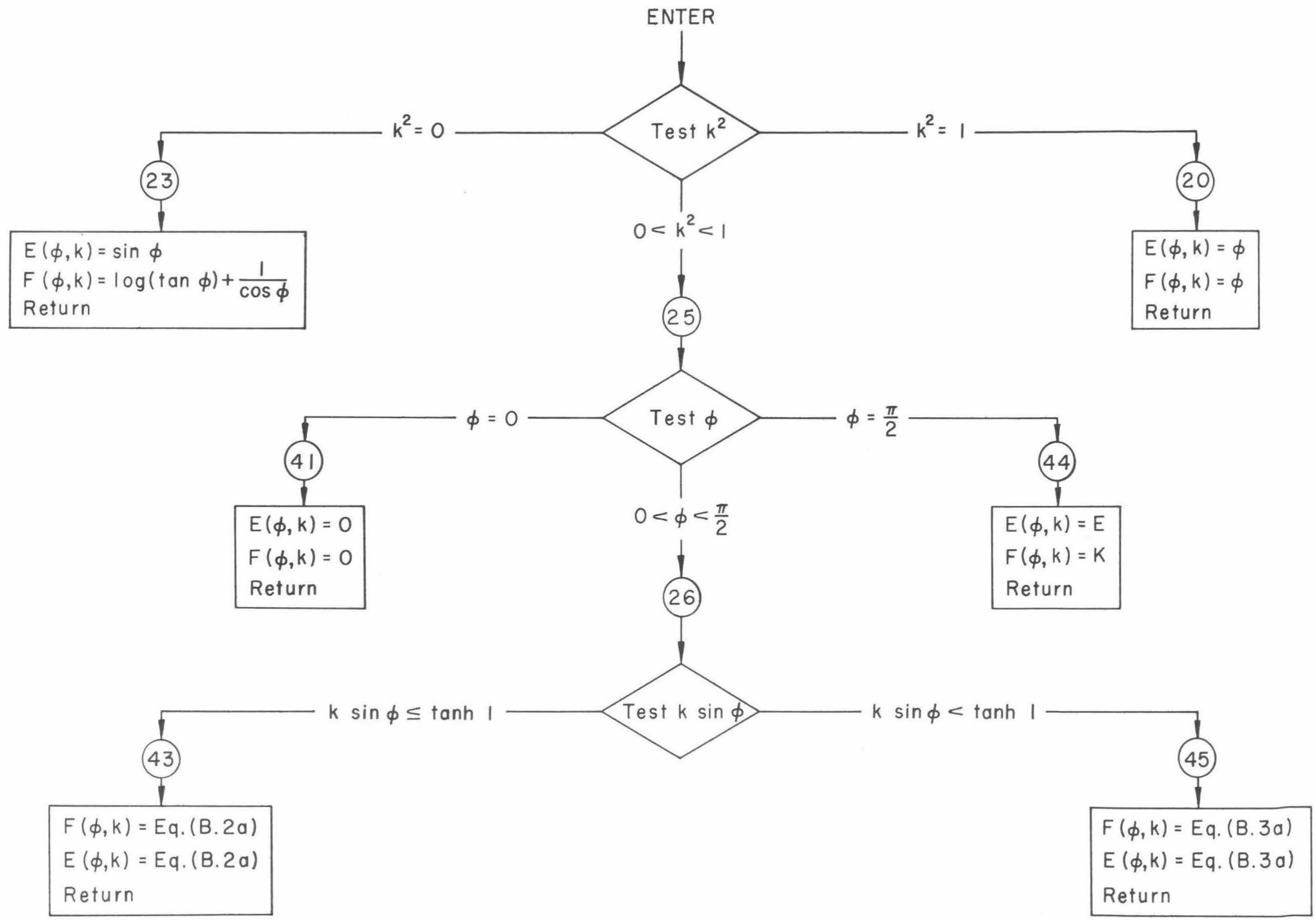

Figure C-4. The Fortran IV listing of Subroutine ELLI.

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